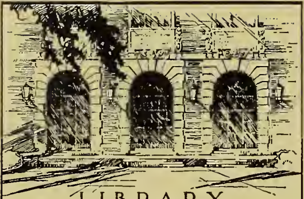

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Technical Report No. 53

A STUDY OF THE NON-UNIFORM CONVERGENCE OF THE INVERSE OF A DOUBLY-INFINITE MATRIX ASSOCIATED WITH A BOUNDARY VALUE PROBLEM IN A WAVEGUIDE

by

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1. INTRODUCTION

In formulating electromagnetic boundary value problems we are often led to an infinite set of equations. In most cases it is not possible to invert the infinite matrix and we are forced to resort to truncating the above to a finite size. We then solve the equations for a number of increasing sizes of the matrix and study the convergence of the solution. If at least the leading members of the unknown coefficients (which are usually of primary interest) tend to converge, we feel satisfied and assume that we have obtained a reasonably good approximation for the leading coefficients.

It is the purpose of this paper to show in the first instance that for a particular doubly infinite set of equations associated with the bifurcation problem in a waveguide, there is a conditional convergence of the solution, meaning that the solution converges to a different set of answers for every different choice of the ratio C = P/Q. P and Q are the numbers of equations from the first and second set, respectively, out of the doubly infinite set of equations. These equations for the bifurcation problem have been obtained by Hurd and Gruenberg[1], who have also presented an exact solution of the infinite set through the use of calculus of residues. The above set, because it has a known solution, is particularly suited for our purpose which is to demonstrate by inverting several finite size matrices that there results a conditional convergence when $P \rightarrow \infty$ and $Q \rightarrow \infty$. Furthermore, we are also able to find the ratio of P/Q which yeilds the correct answer in the limit.

In the second part of this paper we present a theoretical basis for choosing the correct ratio of P/Q when working with a truncated set. With this choice, when the size of the set is increased indefinitely while keeping the ratio C constant, the solution does converge to the correct answer. It is shown through the study of the asymptotic behavior of the higher order unknown coefficients and the application of the edge condition, why only an unique choice of the ratio would make the solution asymptotically tend to the correct one and why otherwise an incorrect solution will result.

In a separate paper to be published in the near future we shall apply the ideas developed here to other problems in a waveguide which cannot be solved exactly. We will again show that there is non-uniform convergence of the doubly infinite set and how using the ideas developed here one is able to choose the correct ratio of P/Q for the problem under consideration.

2. DEVELOPMENT OF THE INFINITE SET OF EQUATIONS FOR H-PLANE BIFURCATION

The infinite set of equations which will be discussed here in connection with the problem of H-plane bifurcation in a rectangular waveguide have been derived by Hurd[1]. We shall therefore skip the details and merely outline the procedure for their derivation.

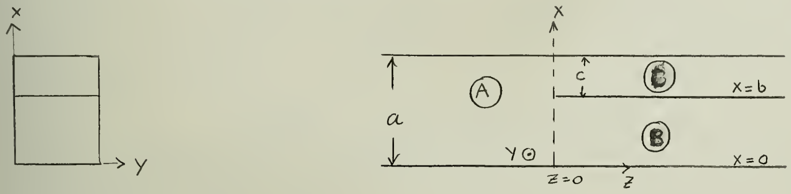

Figure 1. H-Plane Bifurcation of a Rectangular Waveguide

The geometry under consideration is shown in Figure 1. Assume that the incident wave from the negative z-direction is a TE_{10} mode with the electric vector parallel to the edge of the septum. It can be easily shown that the only non-zero field components are E_y, H_x and H_z and that they can all be expressed in terms of a scalar function $\phi = E_y$ and its partial derivatives.

The problem can be stated in terms of the equation

$$\nabla_{xz}^2 \phi + k^2 \phi = 0 \tag{1}$$

and the following boundary conditions on ϕ:

a) ϕ and $\nabla\phi$ are finite everywhere in the region concerned except at the edge of the bifuraction at $z = 0$, $x = b$ where $\nabla\phi$ becomes infinite.

b) ϕ and $\nabla\phi$ are continuous in the subregions and at $z = 0$.

c) ϕ behaves as an outgoing wave at large (z) apart from the incident field.

d) ϕ vanishes on the walls $x = 0$, a and on the septum at $x = b$, $z > 0$.

e) ϕ satisfies the edge condition and hence goes to zero at the edge as $r^{1/2}$ where r is the distance from the edge. $\nabla\phi$ therefore goes to infinite as $r^{-1/2}$ as $r \rightarrow 0$.

It is easily verified that ϕ_A, ϕ_B and ϕ_C expressed in the following equations satisfy the equation (1) and the conditions (a), (c) and (d). The expressions are:

$$\phi_A = A \sin \frac{\pi x}{a} + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} e^{\alpha_n z} \tag{2a}$$

$$\phi_B = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{b} e^{-\beta_n z} \tag{2b}$$

$$\phi_C = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{c} e^{-\gamma_n z} \tag{2c}$$

where

A = amplitude of the incident field

$$\alpha_n = [(\frac{n\pi}{a})^2 - k^2]^{1/2}$$

$$\beta_n = [(\frac{n\pi}{b})^2 - k^2]^{1/2}$$

$$\gamma_n = [(\frac{n\pi}{c})^2 - k^2]^{1/2}$$

$k = \frac{2\pi}{\lambda}$, λ = free space wave length.

α, β and γ's are the mode propagation constants in the three regions.

By applying the continuity conditions at $z = 0$, and subsequently equating the Fourier components of the resulting equations in the range $0 < x < b$ and

$b < x < c$ one arrives at the following doubly infinite set of equations after some manipulations (for details see Hurd):

$$B_p = \frac{(-)^p \frac{2p\pi}{c^2} \sum_{n=1}^{\infty} (A\,\delta'_n + A_n) \sin \frac{n\pi b}{a}}{\alpha_n^2 - \beta_p^2}$$

$$\sum_{n=1}^{\infty} \frac{A_n \sin \frac{n\pi b}{a}}{\alpha_n - \beta_p} = \frac{A \sin \frac{\pi b}{a}}{\alpha_1 + \beta_p} \qquad p = 1, \ldots . \infty \qquad (3a)$$

$$\sum_{n=1}^{\infty} \frac{A_n \sin \frac{n\pi b}{a}}{\alpha_n - \gamma_q} = \frac{A \sin \frac{\pi b}{a}}{\alpha_1 + \gamma_q} \qquad q = 1, \ldots . \infty \qquad (3b)$$

The set of infinite equations (3) has been solved exactly by Hurd. We shall however, concentrate on the solution of the truncated set of equations for various combinations of P and Q where these are the number of equations from the set (3a) and (3b) respectively. We proceed to do this in the following section.

3. SOLUTION OF TRUNCATED SET OF EQUATIONS

Consider the solution of the turncated set of equations

$$\sum_{n=1}^{P+Q} \frac{A_n \sin \frac{n\pi b}{a}}{\alpha_n - \beta_p} = \frac{A \sin \frac{\pi b}{a}}{\alpha_1 + \beta_p} \qquad p = 1,\ 2,\ \ldots.\ P \qquad (4a)$$

$$\sum_{n=1}^{P+Q} \frac{A_n \sin \frac{n\pi b}{a}}{\alpha_n - \gamma_q} = \frac{A \sin \frac{\pi b}{a}}{\alpha_1 + \gamma_q} \qquad q = 1,\ 2,\ \ldots.\ Q \qquad (4b)$$

In attempting to solve the set of equations (4) we recognize first of all that the determinant of the equation is of a particular kind which is called a double alternant. Written explicitly the determinant is

$$\Delta \doteq \begin{vmatrix} \frac{1}{\alpha_1-\beta_1}, & \frac{1}{\alpha_2-\beta_1}, & \frac{1}{\alpha_3-\beta_1}, & \ldots.., & \frac{1}{\alpha_{P+Q}-\beta_1} \\ \frac{1}{\alpha_1-\beta_2}, & \frac{1}{\alpha_2-\beta_2}, & \frac{1}{\alpha_3-\beta_2}, & \ldots.., & \frac{1}{\alpha_{P+Q}-\beta_2} \\ - - - & - - - & - - - & - - - & - - - \\ \frac{1}{\alpha_1-\beta_P}, & \frac{1}{\alpha_2-\beta_P}, & \frac{1}{\alpha_3-\beta_P}, & \ldots.., & \frac{1}{\alpha_{P+Q}-\beta_P} \\ \frac{1}{\alpha_1-\gamma_1}, & \frac{1}{\alpha_2-\gamma_1}, & \ldots.., & \ldots.., & \frac{1}{\alpha_{P+Q}-\gamma_1} \\ & & & & \\ \frac{1}{\alpha_1-\gamma_Q}, & \frac{1}{\alpha_2-\gamma_Q}, & \ldots..., & \ldots.., & \frac{1}{\alpha_{P+Q}-\gamma_Q} \end{vmatrix} \qquad (5)$$

It is to be noted that we have grouped the factors $\sin n\pi b/a$ with the unknowns A_n.

It is observed that each element of the determinant is a reciprocal of the difference of two quantities, α_r and β_s or γ_t in general, only one of which, vi the α, changes as one goes along the columns whereas only β or γ changes as one goes down the rows. Hence if $\alpha_r \to \alpha_n$, where r and n are two different subscripts, then it is obvious that the determinant $\Delta \to 0$ and that the zero is a simple one. This is effective to saying that $(\alpha_r - \alpha_n)$ for various combinations of r and n are the factors in the continued product expansion of the determinant. In a similar manner we observe that $(\beta_r - \beta_n)$, $(\gamma_r - \gamma_n)$ and $(\beta_t - \gamma_s)$ are also to be included in the product expansion of the determinant. We have to make sure, however, that all such factors are included in the product and that none is repeated. It is also obvious that the denominator of this expression for Δ must contain the factors $(\alpha_t - \beta_s)$ and $(\alpha_t - \gamma_r)$ for various combinations of the subscripts. The expression for the Δ developed on the basis of above arguments, is

$$\Delta = (-1)^{\nu} \frac{\prod\limits_{m=(n+1),n=1}^{M,M-1} (\alpha_m - \alpha_n) \prod\limits_{m=n+1,n=1}^{P,P-1} (\beta_m - \beta_n) \prod\limits_{m=n+1,n=1}^{Q,Q-1} (\gamma_m - \gamma_n) \prod\limits_{m=1,n=1}^{Q,P} (\beta_m - \gamma_n)}{\prod\limits_{m=1,n=1}^{M,P} (\alpha_m - \beta_n) \prod\limits_{m=1,n=1}^{M,Q} (\alpha_m - \gamma_n)} \tag{6}$$

where M = P + Q; ν is an even or odd integer depending on P and Q and as yet undertermined, although we shall not really need to find ν. This is because in order to calculate the coefficients A_r we will have to find only the ratio of the determinants Δ_r/Δ where Δ_r is the determinant obtained by replacing the rth column of Δ by the column representing the right hand side of (4). It is fairly straightforward to see that

$$\Delta_r = A \;\; \Delta(\alpha_r \to -\alpha_1) \tag{7}$$

where $\Delta(\alpha_r \to -\alpha_1)$ is the determinant Δ with α_r replaced by $-\alpha_1$. We can therefore obtain through the use of (6) and (7) an expression for A_r in the product form, which is

$$A_r \frac{\sin\left(\frac{r\pi b}{a}\right)}{\sin\left(\frac{\pi b}{a}\right)} = A \frac{\Delta(\alpha_r \rightarrow -\alpha_1)}{\Delta}$$

$$= A \prod_{n=1}^{P} \frac{(\alpha_r - \beta_n)}{(-\alpha_1 - \beta_n)} \prod_{n=1}^{Q} \frac{(\alpha_r - \gamma_n)}{(-\alpha_1 - \gamma_n)} \cdot \prod_{n=1}^{r-1} \frac{(-\alpha_1 - \alpha_n)}{(\alpha_r - \alpha_n)} \prod_{n=1}^{M} \frac{(\alpha_n + \alpha_1)}{(\alpha_n - \alpha_r)}$$

$$= (-)^{P+Q} A \prod_{n=1}^{P} \frac{(\alpha_r - \beta_n)}{(\alpha_1 + \beta_n)} \prod_{n=1}^{Q} \frac{(\alpha_r - \gamma_n)}{(\alpha_1 + \gamma_n)} \prod_{n=1}^{M\,(1)} \frac{(\alpha_n + \alpha_1)}{(\alpha_n - \alpha_r)} \tag{8}$$

where the superscript (1) on the last product implies that the factor corresponding to n = r is to be omitted. Equation (8) gives the desired solution for the truncated matrix. In particular the expression for A_1/A, which can be identified with the reflection coefficient R is

$$R = \frac{A_1}{A} = (-)^{P+Q} \prod_{n=1}^{P} \frac{(\alpha_1 - \beta_n)}{(\alpha_1 + \beta_n)} \prod_{n=1}^{Q} \frac{(\alpha_1 - \gamma_n)}{(\alpha_1 + \gamma_n)} \prod_{n=2}^{M} \frac{(\alpha_n + \alpha_1)}{(\alpha_n - \alpha_1)}$$

$$= \prod_{n=1}^{P} \frac{\left(1 - \frac{\alpha_1}{\beta_n}\right)}{\left(1 + \frac{\alpha_1}{\beta_n}\right)} \prod_{n=1}^{Q} \frac{\left(1 - \frac{\alpha_1}{\gamma_n}\right)}{\left(1 + \frac{\alpha_1}{\gamma_n}\right)} \prod_{n=2}^{M} \frac{\left(1 + \frac{\alpha_1}{\alpha_n}\right)}{\left(1 - \frac{\alpha_1}{\alpha_n}\right)} \tag{9}$$

We also quote below for comparison and reference, the result arrived at by Hurd as a solution of the infinite set of equations. His expression is

$$R = \frac{\Pi^{(1)}(-\alpha_1,\alpha)\ \Pi(\alpha_1,\beta)\ \Pi(\alpha_1,\gamma)}{\Pi^{(1)}(\alpha_1,\alpha)\ \Pi(-\alpha_1,\beta)\ \Pi(-\alpha_1,\gamma)}$$

$$\times \exp\left[(2\alpha_1/\pi)\left\{-a + b\ln(a/b) + c\ln(a/c)\right\}\right] \tag{10}$$

where for instance

$$\Pi(\omega,\alpha) = \prod_{p=1}^{\infty} (\alpha_p - \omega)\left(\frac{a}{p\pi}\right) e^{\frac{a\omega}{p\pi}}$$

and the superscript (1) implies as before that the factor corresponding to $p = 1$ is to be omitted.

4. NUMERICAL CALCULATIONS OF SOLUTIONS OF FINITE AND INFINITE SET OF EQUATIONS

In this section we shall present and compare the numerical values of the reflection coefficient R for certain choices of b/c, b/a (note these fix c/a) γ etc. We shall see that even when (P + Q) is very large the expression (9) for a reflection coefficient yields different values for different ratios of P/Q. We shall then compare the answer for a particular choice of P/Q and see that it indeed converges to the exact value of R calculated through the use of (10).

For the purposes of numerical calculation we shall consider the particular case when all the mode propagation constants excepting α_1 are real and α_1 is purely imaginary. This is done merely for the ease of numerical work, and the conclusions regarding the convergence phenomenon reached in this case will still be applicable to the general case when the above condition is not true.

For this case it is obvious from (9) that $R = e^{j\theta}$, i.e. $|R| = 1$. The angle θ = argument R is given by

$$\theta = \arg R = -2\left[\sum_{n=1}^{P} \tan^{-1}\frac{\delta}{\beta_n} + \sum_{n=1}^{Q} \tan^{-1}\frac{\delta}{\gamma_n} - \sum_{n=1}^{P+Q} \tan^{-1}\frac{\delta}{\alpha_n}\right] \qquad (11)$$

where $\alpha_1 = j\delta$

But since $\beta_n = \sqrt{(n\pi/b)^2 - k^2}$ and $\delta = \sqrt{k^2\ (\pi/a)^2}$, one can write

$$\tan^{-1}\frac{\delta}{\beta_n} = \sin^{-1}\frac{\frac{b\delta}{\pi}}{\left(n^2 - \frac{b^2}{a^2}\right)^{1/2}}$$

After similarly expressing the other arctans in terms of arc sines, one can rewrite (11) as

$$\frac{\theta}{2} = \sum_{n=1}^{P+Q} \sin^{-1} \frac{\frac{a\delta}{\pi}}{\sqrt{n^2 - 1}} - \sum_{n=1}^{P} \sin^{-1} \frac{\frac{b\delta}{\pi}}{\sqrt{n^2 - (\frac{b}{n})^2}} - \sum_{n=1}^{Q} \sin^{-1} \frac{\frac{c\delta}{\pi}}{\sqrt{n^2 - (\frac{c}{a})^2}} \tag{12}$$

For the purposes of numerical calculation (12) is more suitable than (8) A similar expression for the exact θ_{ex} has been derived from (10) and is given in the following

$$\frac{\theta_{ex}}{2} = - \frac{\delta}{\pi} (1 - \frac{b}{a} \ell n \frac{a}{b} - \frac{c}{a} \ell n \frac{a}{c}) + S_2 (\frac{\delta}{\pi} ; 1, 0)$$

$$- S_1 (\frac{\delta b}{a} ; \frac{b}{a} , 0) - S_1 (\frac{\delta c}{\pi} ; \frac{c}{a} , 0) \tag{13}$$

where

$$S_N (u ; \nu , 0) = \sum_{n=N}^{\infty} [\sin^{-1} \frac{u}{(n^2 - b^2)^{1/2}} - \frac{u}{n}].$$

Calculations were done for $\theta/2$ for the following of dimensions of the waveguide bifurcation

$a = 2.286$ $b/a = 0.313$ hence $c/a = 0.687$ $b/c = 0.456$

and for $\lambda = 3$ cms: The results for θ, calculated using (11) are shown in the following table for various combinations for P and Q and are calculated to the nearest degree. The θ values are compared with θ_{ex} calculated from (13).

It is seen that the values of θ converge to different numbers for different values of P/Q but that the value of θ is not too sensitive for slight changes in P/Q as may be seen from the last four lines of the table in which θ is seen to agree with θ_{ex}. It is also observed that there is a considerable deviation from the correct value for some choices of P/Q. The correct choice for P/Q

TABLE 1

P	Q	P/Q	θ (radians) (calculated)	θ_{ex} (radians) (exact)
10	10	1	-42°	-50°
20	20	1	-42°	
20	10	2	-18°	
30	15	2	-18°	
8	17	0.471	-50°	
10	22	0.455	-50°	
20	44	0.455	-50°	
10	20	0.5	-50°	

seems to be in the vicinity of P/Q = 0.48 and to find it more accurately through numerical means one has to calculate the results to a higher degree of accuracy. In the following section we give an explanation for the behavior of the results displayed in Table 1 and a theoretical basis for the correct choice of P/Q. It will be shown that the criterion for the proper choice of P/Q is based on the asymptotic behavior of the higher order mode coefficients A_r, for (P + Q) large. We develop the necessary formulas for the study of the asymptotic behaviors in the following section.

5. ASYMPTOTIC BEHAVIOR OF HIGHER ORDER COEFFICIENTS

In this section starting from (8) we shall develop an asymptotic expression for A_r/A for large P and Q.

Let us rewrite (8) as

$$A_r' = \prod_{n=1}^{P} \frac{\left(1 - \frac{\alpha_r}{\beta_n}\right)}{\left(1 + \frac{\alpha_1}{\beta_n}\right)} \prod_{n=1}^{Q} \frac{\left(1 - \frac{\alpha_r}{\gamma_n}\right)}{\left(1 + \frac{\alpha_1}{\gamma_n}\right)} \prod_{n=1}^{P+Q}{}^{(1)} \frac{\left(1 + \frac{\alpha_1}{\alpha_n}\right)}{\left(1 - \frac{\alpha_r}{\alpha_n}\right)} \tag{14}$$

where

$$A_r' = A_r \frac{\sin \frac{r\pi b}{a}}{\sin \frac{\pi b}{a}}$$

Since

$$\beta_n \to \frac{n\pi}{b}, \qquad \gamma_n \to \frac{n\pi}{c} \qquad \text{and} \qquad \alpha_n \to \frac{n\pi}{a} \qquad \text{for large } n,$$

it is convenient to introduce some additional factors in the numerator and denominator of (14) and rearrange it as

$$A_r' = \left\{ \prod_{n=1}^{P} \frac{\left(1 - \frac{\alpha_r}{\beta_n}\right)}{\left(1 - \frac{\frac{\alpha_r b}{\pi}}{n}\right)} \prod_{n=1}^{Q} \frac{\left(1 - \frac{\alpha_r}{\gamma_n}\right)}{\left(1 - \frac{\frac{\alpha_r c}{\pi}}{n}\right)} \prod_{n=1}^{P+Q}{}^{(r)} \frac{\left(1 - \frac{\frac{\alpha_r a}{\pi}}{n}\right)}{\left(1 - \frac{\alpha_r}{\alpha_n}\right)} \right\}$$

$$\times \left\{ \prod_{n=1}^{P} \frac{\left(1 + \frac{\frac{\alpha_1 b}{\pi}}{n}\right)}{\left(1 + \frac{\alpha_1}{\beta_n}\right)} \prod_{n=1}^{Q} \frac{\left(1 + \frac{\frac{\alpha_1 c}{\pi}}{n}\right)}{\left(1 + \frac{\alpha_1}{\gamma_n}\right)} \prod_{n=1}^{P+Q} \frac{\left(1 + \frac{\alpha_1}{\gamma_n}\right)}{\left(1 + \frac{\frac{\alpha_1 a}{\pi}}{n}\right)} \right\}$$

$$\times \frac{\left(1 - \frac{\frac{\alpha_r a}{\pi}}{r}\right)}{\left(1 - \frac{\frac{\alpha_1 a}{\pi}}{r}\right)} \frac{\prod_{n=1}^{P} \left(1 - \frac{\frac{\alpha_r b}{\pi}}{n}\right) \prod_{n=1}^{Q} \left(1 - \frac{\frac{\alpha_r c}{\pi}}{n}\right)}{\prod_{n=1}^{P+Q} \left(1 - \frac{\frac{\alpha_r a}{\pi}}{n}\right)} \cdot \frac{\prod_{n=1}^{P+Q} \left(1 + \frac{\frac{\alpha_1 a}{\pi}}{n}\right)}{\prod_{n=1}^{P} \left(1 + \frac{\frac{\alpha_1 b}{\pi}}{n}\right) \prod_{n=1}^{Q} \left(1 + \frac{\frac{\alpha_1 c}{\pi}}{n}\right)}$$

Now notice first of all that the factors inside the first two curly brackets tend asymptotically to constants as P and Q are increased indefinitely for a given r. This is because the factors like

$$\frac{\left(1 - \dfrac{\frac{a_r a}{\pi}}{n}\right)}{\left(1 - \dfrac{a_r}{a_n}\right)} \longrightarrow 1 \quad \text{for large } n,$$

and similar reasoning holds for other factors appearing inside the curly brackets.

It will therefore be sufficient to study the behavior of the ratio of the products F, where

$$F = \frac{\prod_{n=1}^{P} \left(1 - \dfrac{\frac{a_r b}{\pi}}{n}\right) \prod_{n=1}^{Q} \left(1 - \dfrac{\frac{a_r c}{\pi}}{n}\right)}{\prod_{n=1}^{P+Q} \left(1 - \dfrac{\frac{a_r a}{\pi}}{n}\right)} \left(1 - \dfrac{\frac{a_r a}{\pi}}{r}\right) \qquad (16)$$

and of a similar ratio of products with $-a_1$ replacing a_r.

Let us rewrite (16) as

$$F = \left(1 - \dfrac{\frac{a_r a}{\pi}}{r}\right) \frac{P^{a_r b/\pi} \prod_{n=1}^{P} \left(1 - \dfrac{\frac{a_r b}{\pi}}{n}\right) Q^{a_r c/\pi} \prod_{n=1}^{Q} \left(1 - \dfrac{\frac{a_r c}{\pi}}{n}\right)}{(P+Q)^{a_r b/\pi} \prod_{n=1}^{P+Q} \left(1 - \dfrac{\frac{a_r a}{\pi}}{n}\right)} \; \frac{(P+Q)^{a_r a/\pi}}{P^{a_r b/\pi} \; Q^{a_r c/\pi}} \qquad (17)$$

nd then use the following representation (see Magnus[2] page 2) of $1/\Gamma(z+1)$ here $\Gamma(x)$ is the Gamma function of argument x.

$$\frac{1}{\Gamma(z+1)} = \lim_{m \to \infty} m^{-z} \prod_{n=1}^{m} \left(1 + \frac{z}{n}\right) \tag{18}$$

to recast (16) into its asymptotic form for large P and Q. We derive using (18) in (17), and letting P/Q = C(constant)

$$\lim_{\substack{P \to \infty \\ Q \to \infty}} F = \left(1 - \frac{\frac{a_r a}{\pi}}{r}\right) \frac{\Gamma\left(1 - \frac{a_r a}{\pi}\right)}{\Gamma\left(1 - \frac{a_r b}{\pi}\right) \Gamma\left(1 - \frac{a_r c}{\pi}\right)} \cdot \left(1 + \frac{1}{C}\right)^{a_r b/\pi} (1 + C)^{a_r c/\pi} \tag{19}$$

Next using (see Magnus[2] page 1)

$$\Gamma(1 - z) = \frac{\pi}{\sin \pi z \quad \Gamma(z)}$$

in (19) one derives

$$\lim_{\substack{P \to \infty \\ Q \to \infty}} F = \frac{(1 - \frac{a_r a/\pi}{r}}{\pi}) \cdot \frac{\sin(a_r b) \sin(a_r c)}{\sin a_r a} \cdot \frac{\Gamma(a_r b/\pi) \Gamma(a_r c/\pi)}{\Gamma\left(\frac{a_r a}{\pi}\right)}$$

$$\times \left(1 + \frac{1}{C}\right)^{a_r b/\pi} (1 + C)^{a_r c/\pi} \tag{20}$$

The next step is to study the limit of F as r and hence a_r becomes very large. We have from Stirling's formula,

$$\Gamma(z) \approx \frac{z^z}{z^{1/2}} \qquad \text{for a large } z \tag{21}$$

Using (21) in (20) and letting $a_r \to r\pi/a$ we find

$$F \to K \frac{\left(\frac{rb}{a}\right)^{rb/a} \left(\frac{rc}{a}\right)^{rc/a}}{(r)^r} \cdot \frac{1}{r^{3/2}} \left(1 + \frac{1}{C}\right)^{rb/a} (1 + C)^{rc/a}$$

for large r, P and Q,

(22)

where K is a function with a limited upper bound for arbitrarily large r. It is to be noted that we have used the fact that $(1 - (\alpha_r a/\pi)/r) \big/ \sin \alpha_r a$ has the limit $r\pi/a$ as $\alpha_r \to r\pi/a$.

Writing

$$r^r = \left(\frac{rb}{a} + \frac{rc}{a}\right)^{rb/a + rc/a}$$

we obtain from (22)

$$F \to \frac{K}{r^{3/2}} \; \frac{\left(1 + \frac{1}{C}\right)^{rb/a} \left(1 + C\right)^{rc/a}}{\left(1 + \frac{c}{b}\right)^{rb/a} \left(1 + \frac{b}{c}\right)^{rc/a}} \tag{23}$$

for $P/Q = C$, $P \to \infty$, $Q \to \infty$ and r large.

Equation (23) gives the desired asymptotic behavior of F. It is not ifficult to show that the ratio of products

$$\frac{\displaystyle\prod_{n=1}^{P+Q} \left(1 + \frac{\frac{\alpha_1 a}{\pi}}{n}\right) \cdot \frac{1}{\left(1 - \frac{\frac{\alpha_1 a}{\pi}}{r}\right)}}{\displaystyle\prod_{n=1}^{P} \left(1 + \frac{\frac{\alpha_1 b}{\pi}}{n}\right) \prod_{n=1}^{Q} \left(1 + \frac{\frac{\alpha_1 c}{\pi}}{n}\right)} = G, \quad \text{say,}$$

has the asymptotic behavior

$$\lim_{\substack{P \to \infty \\ Q \to \infty}} G = \frac{1}{\left(1 - \frac{\frac{\alpha_1 a}{\pi}}{r}\right)} \cdot \frac{\Gamma\left(1 + \frac{\alpha_1 b}{\pi}\right) \Gamma\left(1 + \frac{\alpha_1 c}{\pi}\right)}{\Gamma\left(1 + \frac{\alpha_1 a}{\pi}\right)} \left(1 + \frac{1}{C}\right)^{-\alpha_1 b/\pi} \left(1 + C\right)^{-\alpha_1 c/\pi} \tag{24}$$

and hence tends to a constant as r becomes large. We can sum all this up and arrive at the limit

$$\lim_{\substack{P \to \infty \\ Q \to \infty}} A'_r \sim \frac{K_1}{r^{3/2}} \cdot \frac{(1 + \frac{1}{C})^{rb/a} \ (1 + C)^{rc/a}}{(1 + \frac{c}{b})^{rb/a} \ (1 + \frac{b}{c})^{rc/a}} \quad \text{for large } r, \tag{25}$$

where K_1 is a constant.

Suppose now that we pick a C which is different than the ratio of b/c and let $C \geq b/c$. Calling $X = b/c$, we can write (25) after some manipulation as

$$\lim_{\substack{P \to \infty \\ Q \to \infty}} A'_r \sim \frac{K_1}{r^{3/2}} \left\{ \frac{(1 + C)}{(1 + X)} \cdot \left(\frac{X}{C}\right)^{X} \right\}^{rc/a} \quad \text{for large } r. \tag{26}$$

Without loss of generality we can let X (= b/c) > 1, and show that

$$C^X + C^X X > X^X + X^X C, \qquad \text{since} \qquad C > X > 1$$

and hence

$$\left(\frac{C}{X}\right)^X > \left(\frac{1 + C}{1 + X}\right) \qquad \text{or} \qquad \left(\frac{1 + C}{1 + X}\right)\left(\frac{X}{C}\right)^X < 1 \tag{27}$$

In view of (27), (26) may be written as

$$\lim_{\substack{P \to \infty \\ Q \to \infty}} A'_r \sim \frac{K_1}{r^{3/2}} \eta^r \quad \text{for large r and} \quad C \geq X > 1, \tag{28}$$

where $\eta < 1$.

This shows that under this condition the higher order coefficients have an exponential decay with r unless $\eta = 1$, which happens when $C = X$.

Similarly it may be shown that if $C \leq X > 1$,

$$\lim_{\substack{P \to \infty \\ Q \to \infty}} A'_r \sim \frac{K_1}{r^{3/2}} \zeta^r \qquad \text{for large } r$$

where $\zeta > 1$, i.e., that in this case the coefficient A'_r has an exponential growth with r unless h = 1, which happens again when C = X.

When C = X, then the coefficients A_r have an algebraic behavior for large r and go to zero as $1/r^{3/2}$.

This completes the study of the asymptotic behavior of the higher order coefficients for different ranges of C/X with X > 1. In summarizing we note:

(i) A'_r has an exponential growth for large r when X > C and an exponential decay for C > X.

(ii) A'_r has an algebraic behavior and is $O(1/r^{3/2})$ for large r when C = X.

In the next section we appeal to the edge condition given by condition (e) in Section 2 and show how the proper choice for C = P/Q is to be made.

6. THE EDGE CONDITION AND THE PROPER CHOICE OF C

The condition at the edge of septum requires that the field potential ϕ go to zero at the edge as $r^{1/2}$ and that $|\nabla\phi|$ go to infinity as $r^{-1/2}$. Now it is seen from (2a) that

$$\left.\frac{\partial\phi_A}{\partial z}\right|_{x=b} = A \sin\frac{\pi b}{a} + \sum_{n=1}^{\infty} (A_n \sin\frac{n\pi b}{a}) \cdot \frac{n\pi}{a} e^{\alpha_n z} \tag{29}$$

Hence it is obvious that if $A'_n = \sin(n\pi b/a)A_n/\sin(\pi b/a)$ goes to zero exponentailly for large n, which it does when $P/Q = C > X = b/c$, the expression for $\partial\phi A/\partial z|_{x=b}$ at the edge $z = 0$ has a bounded sum and hence violates the edge condition. Also when $C < X$, A'_n has an exponential growth and hence when substituted in (29), makes $\partial\phi A/\partial z$ go to infinity in a much stronger manner than $z^{-1/2}$ as $z \to 0$. It is only when $C = X$ and A'_n is $0(1/n^{3/2})$ for large n that the sum of the series is $0(z^{-1/2})$ as $z \to 0$ as may be shown by following a method due to Hurd[1]. It may be pointed out that in the exact solution obtained by Hurd A'_n is $0(1/n^{3/2})$ for large n, as it of course must.

The conclusion is then that the proper choice of $C = P/Q$ is $C = b/c = X$; only when such a choice is made and, P and Q are increased indifinitely, that the solution converges to the one which satisfies the edge condition, and thus yields an unique answer satisfying the physical condition.

This section will be concluded with one further remark which concerns the coefficients A_m, $m = 1.2\ldots$ etc., m finite. From (15), (16) and (19) it is clear that even when $P \to \infty$ and $Q \to \infty$ the limits of A_1, A_2 etc. are dependent on C, i.e., on the choice of the ratio of P and Q. Numerical calculations for A_1 for various values of C have been made using the above equations and the results agree with those presented in Table 1. Only the choice of $C = b/c$ yields the correct answer for A_1 although the result for the leading coefficient A_1 is not very sensitive in the vicinity of the correct choice of C.

7. FURTHER WORK

Further work has been done along this line in connection with the step discontinuity problem in a waveguide. Although the exact solution of the step problem is not possible, the correct choice of P/Q can be made on the basis of asymptotic behavior of the higher order coefficients. A detailed discussion on this problem will be the subject of a later report.

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AN INVESTIGATION OF THE NEAR FIELDS ON THE CONICAL EQUIANGULAR SPIRAL ANTENNA

by

O. L. McClelland

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